ELSE HØYRUP

ON TORUS MAPS AND ALMOST PERIODIC MOVEMENTS

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **38**, 10

Kommissionær: Munksgaard
København 1972

Notations

S^n: the n-sphere.
E^n: the n-cell.
$T^n = S^1 \times \ldots \times S^1$ (n factors): the n-dimensional torus.
$\sim$ homeomorphic.
$\simeq$ homotopic, bijective equivalent, isomorphic.
$f_\#$, $f^\#$ maps induced by f in homology, cohomology.
f_*, f^* maps induced by f in homotopy, cohomotopy.

Synopsis

In this paper I show that the cohomotopy groups of the n-dimensional torus T^n usually are direct sums of homotopy groups of spheres. Further, I investigate homotopy classification problems of continuous maps from T^n into other topological spaces – especially spaces with "nice" homotopy groups in the lower dimensions. The results are applied to some "reducibility problems" for torus maps and almost periodic movements – in particular I find conditions for almost periodic movements being almost periodically homotopic to periodic movements.

PRINTED IN DENMARK
BIANCO LUNOS BOGTRYKKERI A/S

Introduction

In 1954 H. Tornehave [4] investigated the following problem: In which topological spaces X is every almost periodic movement $x = f(t)$ ($t \in \boldsymbol{R} =]-\infty, \infty[$, $x \in X$) almost periodically homotopic to a periodic movement?

A(lmost) p(eriodic) homotopy between two a.p. movements $f_0(t)$ and $f_1(t)$ means that there exists a uniformly continuous family $f(t,v)$, $v \in [0,1]$, of almost periodic movements starting with $f_0(t)$ and ending with $f_1(t)$.

Let $\mathscr{C}$ denote the class of metric spaces X which are "continuously locally arcwise connected" (see p. 21). Because every CW complex is "continuously locally arcwise connected", $\mathscr{C}$ includes the class $\mathscr{C}'$ of metrizable CW complexes. $\mathscr{C}'$ includes the class $\mathscr{C}''$ of locally compact polyhedrons. Remark: In this paper I only look at *continuous* maps between topological spaces, though I do not explicitely write continuous everywhere. Nor do I everywhere write that I assume my spaces different from the empty set.

A theorem ([4] p. 28) states that every a.p. movement $\tilde{f}(t)$ in a space $X \in \mathscr{C}$ corresponds to some rationally independent real numbers $(\beta_1, \ldots, \beta_n)$ and a continuous torus map f: $\boldsymbol{R}^n \to X$ (f is periodic in all the variables with the period 2π) in such a way that $\tilde{f}(t)$ is almost periodically homotopic to the almost periodic movement $f(\beta_1 t, \ldots, \beta_n t)$.

A small correction and generalization of Theorem 13 in [4] gives:

Every almost periodic movement in $X \in \mathscr{C}$ is almost periodically homotopic to a periodic movement if and only if for every continuous torus map f (of any dimension) into X there exists a number $N \in \boldsymbol{N}$ so that $f \circ (\times N)$ is homotopic to a torus map into a closed curve in X, where $(\times N)(u_1, \ldots, u_n) = (Nu_1, \ldots, Nu_n)$.

Because of this theorem it is natural to look at the following problem: For which X is every torus map (of any dimension) into X homotopic to a torus map into a closed curve in X? H. Tornehave had some intuitive ideas of how to solve the new problem: he thought it was a necessary (and if all the homotopy groups $\pi_i(X,x_0)$ are trivial for $i > 1$, sufficient) condition that for all $x_0 \in X$ every abelian subgroup of the fundamental group $\pi_1(X,x_0)$ is cyclic. (This is not quite correct). In this paper I shall find a partial solution of a more general problem.

Definition 1. *Let X be a topological space. An n-dimensional torus map f: $T^n \to X$ is called m-reducible iff it is homotopic to a continuous map g through T^m,* i.e. *$T^n \to T^m \to X$, where $1 \leq m < n$ are the only interesting cases.*

Definition 2. *The space X is called n-dimensionally m-reducible iff every n-dimensional torus map into X is m-reducible.*

Definition 3. *The space X is called m-reducible iff every torus map into X (of any dimension) is m-reducible.*

Definition 4. *An a.p. movement $x = \tilde{f}(t)$ in X is said to be of dimension $\leqq n$ iff it is a.p. homotopic to some $x = f(\beta_1 t, \ldots, \beta_n t)$, where $(\beta_1, \ldots, \beta_n)$ are rationally independent real numbers and f: $T^n \to X$ is an n-dimensional torus map.*

Definition 5. *The a.p. movement $x = \tilde{f}(t)$ in X of dimension $\leqq n$ is called a.p. m-reducible iff it is a.p. homotopic to an a.p. movement of dimension $\leq m$.*

Definition 6. *The space X is called a.p. n-dimensionally m-reducible iff every a.p. movement in X of dimension $\leq n$ is a.p. m-reducible.*

Definition 7. *The space X is called a.p. m-reducible iff every a.p. movement in X is a.p. m-reducible.*

We have the following theorem:

Theorem 16. *Let the almost periodic movement $x = \tilde{f}(t)$ in $X \in \mathscr{C}$ be a.p. homotopic to $f(\beta_1 t, \ldots, \beta_n t)$, where f: $T^n \to X$, $(\beta_1, \ldots, \beta_n)$ rationally independent. Then*

$$\tilde{f}(t) \text{ is a.p. } m\text{-reducible iff } f \circ (\times N) \text{ for some } N \text{ is } m\text{-reducible.}$$

I shall further study the following problems:

1) When is a given n-dimensional torus map m-reducible?

1a) When is a given a.p. movement $\tilde{f}(t)$ of dimension $\leq n$, a.p. m-reducible?

2) Which spaces X are n-dimensionally m-reducible?

2a) Which spaces X are a.p. n-dimensionally m-reducible?

3) Which spaces X are m-reducible?
3a) Which spaces X are a.p. m-reducible?

The case $m = 1$ is of course of special interest.

My way through the problems is the following:

First I use obstruction theory in dealing with extension problems and homotopy classification problems of continuous functions from subspaces of T^n into topological spaces. I shall not go beyond the primary obstruction because otherwise the problems get too complicated to be of use for my original problems.

Next I look at the special case $X = S^p$, the p-dimensional sphere. The set of homotopy classes of torus maps $T^n \to S^p$, $\pi^p(T^n)$, is called the p-dimensional cohomotopy set. For some p and n the set $\pi^p(T^n)$ is an abelian group which is called the p-dimensional cohomotopy group of T^n. I compute the cohomotopy groups by means of the homotopy groups of the spheres $\pi_i(S^m,s_0)$. Unfortunately, most of these homotopy groups are not yet known.

The homotopy groups of T^n, on the other hand, are very simple: the fundamental group is free abelian of rank n: $\pi_1(T^n,t_0) \simeq \boldsymbol{Z}^n$, and all the higher homotopy groups $\pi_i(T^n,t_0)$ $(i > 1)$ are zero. – In general we know that $\pi_i(X,x_0)$ $(i > 1)$ is an abelian group, while $\pi_1(X,x_0)$ is a group, but not always abelian.

The results obtained on the homotopy classification of torus maps into X, are applied to the torus reducibility problems and this leads to results on the a.p. reducibility problems. For some special topological spaces X this gives simple results, but for further work on almost periodic movements my method does not seem fruitful because the homotopy classification problems soon become extremely complicated and the torus reducibility problems turn out to be more complicated than the a.p. reducibility problems.

I shall state the principal results in a form independent of the choice of basis point. The corresponding theorems in the paper will be stated only for a fixed basis point.

Theorem 18'. *Let $x = \tilde{f}(t)$ be an almost periodic movement in $X \in \mathscr{C}$ corresponding to the torus map f: $T^n \to X$. If $\pi_2(X,x_0) = \ldots = \pi_n(X,x_0) = 0$ for all $x_0 \in X$, then a necessary and sufficient condition for $\tilde{f}$ to be a.p. m-reducible $(1 \leq m < n)$ is that $f_*\pi_1(T^n,t_0)$, which is a finitely generated abelian subgroup of $\pi_1(X,f(t_0))$, has rank $\leqq m$.*

Theorem 19'. *Let $X \in \mathscr{C}$ and let $\pi_i(X,x_0) = 0$ for all $x_0 \in X$, $i > 1$. Then a necessary and sufficient condition for X to be a.p. m-reducible $(m \geq 1)$ is that for all $x_0 \in X$ every abelian subgroup of $\pi_1(X,x_0)$ has rank $\leqq m$.*

Remark: If $\pi_1(X,x_0)$ is itself abelian, the above condition on $\pi_1(X,x_0)$ is equivalent to: rank $\pi_1(X,x_0) \leq m$.

Theorem 20'. *A necessary and sufficient condition for X to be a.p. 1-reducible is that, for all $x_0 \in X$, every abelian subgroup G of $\pi_1(X,x_0)$ has rank $\leqq 1$ and that, for all $n > 1$, the condition that $f_*,g_*: \pi_1(T^n,t_0) \to \pi_1(X,x_0)$ are conjugate implies that for some natural number N the maps $f \circ (\times N)$, $g \circ (\times N)$ are homotopic.*

Theorem 21'. *Let $X \in \mathscr{C}$ be an H-space (for instance a topological group) with $\pi_1(X,x_0) = \ldots = \pi_{p-1}(X,x_0) = 0$ for all $x_0 \in X$. Then a necessary condition for X to be a.p. m-reducible is that for all $x_0 \in X$ the rank of $\pi_p(X,x_0)$ is $\leqq \binom{m}{p}$.*

Theorem 22'. *Let $X \in \mathscr{C}$ and let for all $x_0 \in X$*

$$\pi_1(X,x_0) = \ldots = \pi_{p-1}(X,x_0) = \pi_{p+1}(X,x_0) = \ldots = \pi_m(X,x_0) = 0;$$

then a necessary condition for X to be a.p. m-reducible is that for all $x_0 \in X$ the rank of $\pi_p(X,x_0)$ is $\leqq \binom{m}{p}$ if $p > 1$, and that every abelian subgroup of $\pi_1(X,x_0)$ has rank $\leqq m$, if $p = 1$.

Chapter 1

The Functions *F*, *G*, and *G*'

Definition. *Let X be a topological space with $x_0 \in X$. Let $n \in \boldsymbol{N}$ and let π be a group (abelian for $n > 1$). I call X an m-space of type (π,n) ($m \in \boldsymbol{N} \cup \{+\infty\}$, $m \geq n$) when X is path-connected and its homotopy groups in dimensions $\leq m$, except $\pi_n(X,x_0)$, are zero, while $\pi_n(X,x_0)$ is isomorphic to π.*

A usual (π,n) space is then the same as my ∞-space of type (π,n).

Examples: T^n is an ∞-space of type $(\boldsymbol{Z}^n,1)$,
S^n is an n-space of type $(\boldsymbol{Z},n)$.

For all $n \in \boldsymbol{N}$ and all groups π (abelian if $n > 1$) there exists a topological space X of type (π,n) ([3] p. 426).

We shall now look at maps $T^n \xrightarrow{f} X$ where X is path-connected and $x_0 \in X$. We know that f is homotopic to a map f': $(T^n,t_0) \to (X,x_0)$. Because we are interested only in the homotopy classes of maps $T^n \to X$ we shall always choose representatives g of the homotopy classes for which $g(t_0) = x_0$, but we do not write this explicitely everywhere.

If we denote the homotopy classes of maps equivalent to f: $T^n \to X$ by $[f]$ and the set of these classes by $[T^n,X]$, then we know that a function

$$F_n\colon [T^n,X] \to \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(X,x_0))\ /\ \sim,$$

where $\sim$ means conjugate, (if $\pi_1(X,x_0)$ is abelian, then this group is isomorphic to $\pi_1(X,x_0)^n$) is defined by:

$$F_n[f] = \{f_*\} \text{ (the conjugacy class of } f_*).$$

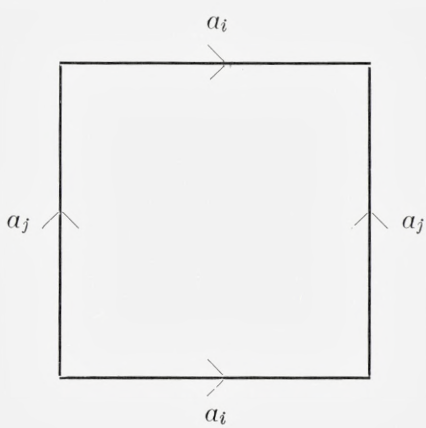
If we denote the homotopy class relative to t_0 of maps $(T^n,t_0) \to (X,x_0)$ equivalent to f by $[f]_{t_0}$ and the set of these maps by $[T^n,t_0;\ X,x_0]_{t_0}$, then we know that $F_n'[f]_{t_0} = f_*$ defines a map

$$F_n'\colon [T^n, t_0; X, x_0]_{t_0} \to \mathrm{Hom}(\pi_1(T^n, t_0),\ \pi_1(X, x_0)).$$

If X is an H-space, we know that $[T^n,X]$ and $[T^n,t_0;\ X,x_0]_{t_0}$ are groups and that $\pi_1(X,x_0)$ is abelian. In this case it is easy to see that F_n and F_n' are homomorphisms.

As a cell complex $T^n = S_1^1 \times \ldots \times S_n^1$ consists of
1 0-dimensional cell t_0,
n 1-dimensional cells S_i^1 (circles), . . . ,
$\binom{n}{p}$ p-dimensional cells $T_i^p = S_{i_1}^1 \times \ldots \times S_{i_p}^1$, . . . ,
and 1 n-dimensional cell T^n.

Let L be a subcomplex of T^n, and let $i_*\colon \pi_1(L,t_0) \to \pi_1(T^n,t_0)$ be induced by the inclusion map $i\colon L \to T^n$. Because $\pi_1(T^n,t_0) \simeq \pi_1(S_1^1,t_0) \oplus \ldots \oplus \pi_1(S_n^1,t_0) \simeq \boldsymbol{Z}^n$ we have that $i_*\pi_1(L,t_0)$ is a direct summand in $\pi_1(T^n,t_0)$ with rank equal to the number r of 1-cells in L. We also have $\pi_1(L,t_0) \simeq \boldsymbol{Z}a_1 * \ldots * \boldsymbol{Z}a_r/\sim$, where $*$ denotes free product and $\sim$ means that two generators a_i and a_j commute when the corresponding 1-cells in L are sides of a 2-cell in L.



A homomorphism $h \in \mathrm{Hom}(\pi_1(L,t_0),\ \pi_1(X,x_0))$ is said to be extensible over T^n iff there exists an $\tilde{h} \in \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(X,x_0))$ with $\tilde{h}i_* = h$.

The statements in the following theorem are proved in [2], VI.

Theorem 1. *Let X be an m-space of type $(\pi,1)$. Then F_n and $F_n{}'$ are surjective if $m \geqq n$-1, and bijective if $m \geqq n$. Let $L \neq \emptyset$ be a subcomplex of T^n and let $g\colon L \to X$ be a given map. Then the following 3 statements are equivalent for $m \geq n$-1:*

(i) *g is continuously extensible over T^n;*
(ii) *$g_* \in \mathrm{Hom}(\pi_1(L,t_0),\ \pi_1(X,x_0))$ is extensible over T^n;*
(iii) *$g_*(\pi_1(L,t_0))$ abelian.*

If g_* is extensible to $h \in \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(X,x_0))$, then we can find an extension f of g with $f_* = h$.

Examples. Let P^m denote the m-dimensional real projective space. Because $\pi_1(P^m,p_0) \simeq \boldsymbol{Z}_2$ for $m \geqq 2$ and $\pi_i(P^m,p_0) = 0$ for $1 < i < m$, while $\pi_m(P^m,p_0) \simeq \boldsymbol{Z}$, we have the bijections

$$[T^n,P^m] \simeq \boldsymbol{Z}_2^n \simeq [T^n,t_0;\ P^m,p_0]_{t_0} \text{ for } n < m.$$

We also have

$$[T^n,S^m] \simeq 0 \simeq [T^n,t_0;\ S^m,s_0]_{t_0} \text{ for } n < m,$$

and

$$[T^n,T^m] \simeq \boldsymbol{Z}^{m \cdot n} \simeq [T^n,t_0;\ T^n,t_0]_{t_0},$$

where the two last bijections are group isomorphisms.

Remark. It is easy to see that F_n surjective (injective) implies F_m surjective (injective) if $m \leqq n$.

Let $(T^n)^m$ denote the m-dimensional skeleton of T^n as a cell-complex. The absolute and relative (with respect to subcomplexes) homology- and cohomology groups of T^n and $(T^n)^m$ are easily computed: With coefficients in an abelian group G we have

$$H_p(T^n;\ G) \simeq G^{\binom{n}{p}} \simeq H^p(T^n;\ G),$$

$$H_p(T^n,L;\ G) \simeq G^{\binom{n}{p}-i(p)} \simeq H^p(T^n,L;\ G),$$

$$H_p((T^n)^m,L';\ G) \simeq G^{\binom{n}{p}-i'(p)} \simeq H^p((T^n)^m,L';\ G),$$

where $i(p)$ and $i'(p)$ denote the number of p-cells in L and L'. As generators for the homology groups we can take the elements $\{\sigma_i \otimes g\}$, $g \in G$, where

σ_i is a p-cell of T^n, $T^n\backslash L$, $(T^n)^m\backslash L'$ respectively corresponding to a p-dimensional torus T_i^p in T^n. As generators for the cohomology groups we can take the elements $\{f_i^g\}$, where f_i^g is the homomorphism determined by $f_i^g(\sigma_i) = g$ and $f_i^g(\sigma_j) = 0$, $j \neq i$.

Further, it is easy to see that $H^p(T^n,\boldsymbol{Z})$ as an algebra is the exterior algebra over $\boldsymbol{Z}$ with n generators in dimension 1 corresponding to the generators of $H^1(S_i^1)$.

The homomorphism

$$h\colon H^p((T^n)^m,L'; G) \to \operatorname{Hom}(H_p((T^n)^m,L'); G)$$

defined by $h\{f\}\{c\} = f(c)$ is an isomorphism. Generators in $\operatorname{Hom}(H_p((T^n)^m, L'); G)$ corresponding to the $\{f_i^g\}$ are the f_i^g themselves.

Dividing the p-cells σ_i into those in $(T^n)^m\backslash$L and those in L I find that

$$H_p((T^n)^m; G) \simeq H_p((T^n)^m,L'; G) \oplus H_p(L',G)$$

$$H^p((T^n)^m; G) \simeq H^p((T^n)^m,L'; G) \oplus H^p(L',G).$$

Thus the homomorphisms

$$i^{\#} \in \operatorname{Hom}(H^p((T^n)^m; G); H^p(L',G)) \text{ and } j^{\#} \in \operatorname{Hom}(H^p((T^n)^m,L'; G), H^p((T^n)^m; G))$$

induced by the inclusion maps $i\colon L' \to (T^n)^m$, $j\colon (T^n)^m \to ((T^n)^m,L')$ are injective, and so are the corresponding homomorphisms between the homology groups $i_{\#}$ and $j_{\#}$.

Now, let $g\colon L' \to X$ be a given map. Suppose $f\colon (T^n)^m \to X$ is an extension of g and denote by $[f]_{L'}$ the homotopy class relative to L' of maps $(T^n)^m \to X$ equivalent to f. By $[(T^n)^m,L',g; X]_{L'}$ we denote the set of these maps. Then there is a well-defined function

$$G'_{n,m,p}\colon [(T^n)^m,L',g; X]_{L'} \to \operatorname{Hom}(H_p((T^n)^m,L'); H_p(X))$$

defined by $G'_{n,m,p}[f]_{L'} = f_{\#p}|H_p((T^n)^m,L')$. We observe that

$$\operatorname{Hom}(H_p((T^n)^m,L'); H_p(X)) \simeq \begin{cases} (H_p(X))^{\binom{n}{p}-i'(p)} & \text{if } p \leqq m \\ 0 & \text{if } p > m. \end{cases}$$

Even if X is an H-space, G' is not in general a homomorphism.

If X is a p-space of type (π_p,p) (π_p abelian), we have the Hurewicz isomorphism $\varkappa\colon \pi_p \to H_p(X)$ and $G'_{n,m,p}$ corresponds to

$$G_{n,m,p}\colon [T^n, L', g; X]_{L'} \to \mathrm{Hom}(H_p((T^n)^m, L'), \pi_p)$$

where $G_{n,m,p}[f]_{L'} = \varkappa^{-1} \circ f_{\#p} \,|H_p((T^n)^m, L')$. We know from the theory of obstruction that $\pi_1(X,x_0) = \ldots = \pi_{p-1}(X,x_0) = 0 \Rightarrow f|(T^n)^{p-1} \simeq 0$, hence we can choose a map $f'\colon (T^n)^m \to X$ with $f' \simeq f$ and $f'|(T^n)^{p-1} = x_0$. If σ denotes a p-cell in $(T^n)^m \setminus L'$, then $f'(\partial\sigma) = x_0$ and $f'|\sigma$ represents an element of $\pi_p \simeq [S^p, X]$. Because $\varkappa([f'|\sigma]) = f'_{\#p}(\sigma) = f_{\#p}(\sigma)$ we have:

$$G_{n,m,p}[f]_{L'}(\sigma) = [f'|\sigma].$$

If $X \neq \emptyset$ is also an H-space, we know that $\pi_1(X,x_0)$ is abelian and that the group structure in $\pi_p(X,x_0)$ is defined by the multiplication map in X, so that in this case the functions G and G' are homomorphisms.

Remark. $G'_n\colon \; [T^n, X] \to \mathrm{Hom}(H_p(T^n),\; H_p(X))$ surjective (injective) implies that $G'_m\colon [T^m, X] \to \mathrm{Hom}(H_p(T^m),\; H_p(X))$ is surjective (injective) for $m \leqq n$.

From the above remarks and the results in [2], VI we get:

Theorem 2. *Let X be an $(m\text{-}1)$-space of type (π_p,p) (π_p abelian). Then we have $[(T^n)^m, L', g; X]_{L'} \neq \emptyset$ and the functions $G_{n,m,p}$ and $G'_{n,m,p}$ are surjective. Let X be an m-space of type (π_p,p) (π_p abelian). Then we have $[(T^n)^m, L', g; X]_{L'} \neq \emptyset$ and $G_{n,m,p}$ and $G'_{n,m,p}$ are bijective.*

Thus

$$[(T^n)^m, L', g; X]_{L'} \simeq \pi_p^{\binom{n}{p} - i'(p)},$$

and, in particular

$$[T^n, L, g;\; X]_L \simeq \pi_p^{\binom{n}{p} - i(p)},$$

$$[T^n, X] \simeq [T^n, t_0;\; X, x_0]_{t_0} \simeq \pi_p^{\binom{n}{p}}.$$

Remark. Let $[(T^n)^m, L', g; X]$ denote the set of homotopy classes of maps $f\colon (T^n)^m \to X$ for which $f|_{L'} = g$. Then $I[f]_{L'} = [f]$ defines a map $[(T^n)^m, L', g; X]_{L'} \to [(T^n)^m, L', g; X]$, which is surjective. The map

$$\tilde{G}'_{n,m,p}\colon \; [(T^n)^m, L', g; X] \to \mathrm{Hom}(H_p((T^n)^m, L'),\; H_p(X)),$$

where $\tilde{G}'_{n,m,p}[f] = f_{\#p}|H_p((T^n)^m, L')$ is also well defined. From this we get ($G'_{n,m,p} = \tilde{G}'_{n,m,p} \circ I$):

$$G'_{n,m,p} \text{ surjective (injective)} \Rightarrow \tilde{G}'_{n,m,p} \text{ surjective (injective)}.$$

Examples. $[(T^n)^m,S^m] \simeq \boldsymbol{Z}^{\binom{n}{m}}$, $[T^n,S^n] \simeq \boldsymbol{Z}$.
(Here $\simeq$ denotes bijection).

Now, let X be an H-space and a p-space of type (π_p,p) (then π_1 is automatically abelian). That X is an H-space means that we have a fixed point $x_0 \in X$, a continuous multiplication μ: $(X\times X,x_0\times x_0) \to (X,x_0)$ for which the constant map $X \xrightarrow{c} x_0$ is a homotopy identity, i.e. $\mu\circ(c,1_x) \simeq \mu\circ(1_x,c) \simeq 1_x$ relative to x_0. From the above theorem we know that for $n \geq p$

$$[(T^n)^{p-1},X] = 0 \text{ and } [(T^n)^p,X] \simeq \mathrm{Hom}(H_p((T^n)^p),\pi_p) \simeq \mathrm{Hom}(H_p(T^n),\pi_p).$$

Hence every $h \in \mathrm{Hom}(H_p(T^n),\pi_p)$ corresponds to a continuous map $(T^n)^p \xrightarrow{f} X$ with $f((T^n)^{p-1}) = x_0$ and $f_{\#p} = \varkappa h$. Let f_i denote the map

$$T^n \xrightarrow{proj} T_i^p \xrightarrow{f|T_i^p} X.$$

Let $\tilde{f}$ denote "one of the products of the f_i's", for instance

$$\mu\circ(1_x\times\mu)\circ\ldots\circ(1_x\times\ldots\times1_x\times\mu)\circ(f_{\binom{n}{p}},\ldots,f_1).$$

Then it is easy to see that $\tilde{f}$ is a continuous map $T^n \to X$ with $\tilde{f}\,|(T^n)^p \underset{(T^n)^{p-1}}{\simeq} f$, i.e. $\tilde{f}_{\#p} = f_{\#p} = \varkappa h$. Thus we have ($n < p$ trivial):

Theorem 3. *Let X be an H-space and a p-space of type (π_p,p). Then for all $n \in \boldsymbol{N}$, G and G' are surjective homomorphisms; G: $[T^n,X] \to \mathrm{Hom}(H_p(T^n),\pi_p) \simeq \pi_p^{\binom{n}{p}}$, where $G([f]) = \varkappa^{-1}f_{\#p}$, G': $[T^n,X] \to \mathrm{Hom}(H_p(T^n),H_p(X)) \simeq \pi_p^{\binom{n}{p}}$, where $G'([f]) = f_{\#p}$.*

For $p = 1$ we have the Hurewicz isomorphism $\varkappa_{T^n}$: $\pi_1(T^n,t_0) \to H_1(T^n)$ and $G([f]) = f_*\circ\varkappa_{T^n}^{-1}$ so that F_n' and F_n are surjective.

Chapter 2

The Cohomotopy Groups of T^n

Definition. *Let X be a topological space and A a subspace of X. The m'th (relative) cohomotopy set $\pi^m(X,A)$ of (X,A) ($m \in \boldsymbol{N}$) is defined to be $[X,A;S^m,s_0]_A$ and the m'th (absolute) cohomotopy set $\pi^m(X)$ of X is defined to be $[X,S^m]$.*

A pair (X,A) is called n-coconnected if it satisfies the condition $H^q(X,A;G) = 0$ for every $q \geqq n$ and every coefficient group G.

A cellular pair (X,A) is a pair of a finite cell complex X and a subcomplex A.

We need the following theorem ([2], VII, Theorem 5.2).

Theorem 4. *If (X,A) is a $(2m\text{-}1)$ coconnected cellular pair, we can define a certain abelian group structure $+$ in $\pi^m(X,A)$ with the class of the constant map as the neutral element. In this case $\pi^m(X,A)$ is called the m'th cohomotopy group of (X,A).*

Remarks. 1) Every map $f\colon (X,A) \to (Y,B)$ induces a transformation $f^*\colon \pi^m(Y,B) \to \pi^m(X,A)$. If both (X,A) and (Y,B) are $(2m\text{-}1)$-coconnected cellular pairs, then f^* is a homomorphism ([2], VII, prop. 5.4).

2) When the cohomotopy group structure is defined in $\pi^m(S^n,s_0)$ (thus $n \leq 2m\text{-}2$), then the bijection $\pi^m(S^n,s_0) \simeq \pi_n(S^m,s_0)$ is an isomorphism ([2], VII, prop. 12.1).

We now try to compute the cohomotopy groups of T^n by means of some exact sequences for the pairs $[(T^n)^m, (T^n)^{m-1}]$.

If $i \geq \,)\frac{m}{2}(\, + 1$, where for α real $)\alpha(= min_{p \in \boldsymbol{Z}}\{p | p \geq \alpha\}$, then $\pi^i((T^n)^m)$ and $\pi^i((T^n)^m,(T^n)^{m-1})$ have the cohomotopy group structure.

From Theorem 2 we have the bijections $(p \geq m)$

$$\pi^p((T^n)^m, (T^n)^{m-1}) \xrightarrow[\simeq]{i^*} \pi^p((T^n)^m) \simeq \begin{cases} 0 & p > m \\ \boldsymbol{Z}^{\binom{n}{m}} & p = m, \end{cases}$$

where i^* is a homomorphism for $p \geq \,)\frac{m}{2}(+ 1$, i.e. $p > 1$.

If X, Y are spaces with basis point, $X \vee Y$ denotes their one point union. If p denotes the map

$$(T^n)^m \to (T^n)^m/(T^n)^{m-1} \sim S^m_1 \vee \ldots \vee S^m_{\binom{n}{m}}$$

then

$$p^*\colon \pi^i(S^m_1 \vee \ldots \vee S^m_{\binom{n}{m}}, s_0) \to \pi^i((T^n)^m, (T^n)^{m-1})$$

is a bijection and for $i \geq \,)\frac{m}{2}(+ 1$ an isomorphism. It is easy to see that there is a bijection

$$\pi^i(S^m_1 \vee \ldots \vee S^m_{\binom{n}{m}}, s_0) \simeq \pi^i(S^m, s_0)^{\binom{n}{m}}.$$

Let i' denote the inclusion $S^m_1 \vee \ldots \vee S^m_r \to S^m_1 \vee \ldots \vee S^m_r \vee \ldots \vee S^m_q$ and

p' the projection $S_1^m \vee \ldots \vee S_q^m \to S_1^m \vee \ldots \vee S_r^m$. When $i \geq)\frac{m}{2}(+ 1$ we get the following split exact sequences of abelian groups and homomorphisms because π^i is a functor:

$$0 \to \pi^i(S_1^m \vee \ldots \vee S_{q-1}^m, s_0) \underset{i'^*}{\overset{p'^*}{\rightleftarrows}} \pi^i(S_1^m \vee \ldots \vee S_q^m, s_0) \underset{p'^*}{\overset{i'^*}{\rightleftarrows}} \pi^i(S_q^m, s_0) \to 0.$$

An induction thus gives us the isomorphism

$$\pi^i(S_1^m \vee \ldots \vee S_q^m, s_0) \simeq \pi^i(S_1^m, s_0) \oplus \ldots \oplus \pi^i(S_q^m, s_0).$$

Because of the isomorphism $\pi^i(S^m, s_0) \simeq \pi_m(S^i, s_0)$ we have that the bijection

$$\pi^i((T^n)^m, (T^n)^{m-1}) \simeq \pi_m(S^i, s_0)^{\binom{n}{m}}$$

is an isomorphism for $i \geq)\frac{m}{2}(+ 1$.

We have the following long exact cohomotopy sequences ([2], VII, 6.–9.) of abelian groups and homomorphisms except the first set and the first map in the first sequence and the two first sets and the two first maps in the second sequence.

1) $m = 2q \geqq 2$.

$$\pi^q((T^n)^{2q-1}) \xrightarrow{\delta_q^{2q}}$$

$$\to \pi^{q+1}((T^n)^{2q}, (T^n)^{2q-1}) \xrightarrow{j_{q+1}^{2q}} \pi^{q+1}((T^n)^{2q}) \xrightarrow{i_{q+1}^{2q}} \pi^{q+1}((T^n)^{2q-1}) \xrightarrow{\delta_{q+1}^{2q}} \ldots$$

$$\to \pi^{2q-1}((T^n)^{2q}, (T^n)^{2q-1}) \xrightarrow{j_{2q-1}^{2q}} \pi^{2q-1}((T^n)^{2q}) \xrightarrow{i_{2q-1}^{2q}} \pi^{2q-1}((T^n)^{2q-1}) \xrightarrow{\delta_{2q-1}^{2q}}$$

$$\to \pi^{2q}((T^n)^{2q}, (T^n)^{2q-1}) \xrightarrow{j_{2q}^{2q}} \pi^{2q}((T^n)^{2q}) \xrightarrow{i_{2q}^{2q}} 0.$$

2) $m = 2q\text{-}1 \geqq 3$.

$$\pi^q((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_q^{2q-1}} \pi^q((T^n)^{2q-1}) \xrightarrow{i_q^{2q-1}} \pi^q((T^n)^{2q-2}) \xrightarrow{\delta_q^{2q-1}}$$

$$\to \pi^{q+1}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{q+1}^{2q-1}} \pi^{q+1}((T^n)^{2q-1}) \xrightarrow{i_{q+1}^{2q-1}} \pi^{q+1}((T^n)^{2q-2}) \xrightarrow{\delta_{q+1}^{2q-1}} \ldots$$

$$\to \pi^{2q-2}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q-2}^{2q-1}} \pi^{2q-2}((T^n)^{2q-1}) \xrightarrow{i_{2q-2}^{2q-1}} \pi^{2q-2}((T^n)^{2q-2}) \xrightarrow{\delta_{2q-2}^{2q-1}}$$

$$\to \pi^{2q-1}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q-1}^{2q-1}} \pi^{2q-1}((T^n)^{2q-1}) \xrightarrow{i_{2q-1}^{2q-1}} 0.$$

All maps i, j are induced by inclusions, and the δ's are connecting "homomorphisms".

We know that every finitely generated abelian group G is isomorphic to a direct sum of cyclic groups. If $(b_1, \ldots, b_n)$ are generators of the cyclic groups in such a decomposition, then they are weakly linearly independent in the sense that $p_1b_1 + \ldots + p_nb_n = 0$ implies $p_1b_1 = \ldots = p_nb_n = 0$. We shall call $(b_1, \ldots, b_n)$ a basis of G.

We now choose basis elements for the finitely generated abelian groups $\pi_j(S^i, s_0)$ represented by $g^{\alpha}_{j,i}$. A generator of $\pi^j(T^j)$ is represented by the projection $p_j\colon T^j \to T^j/(T^j)^{j-1} \sim S^j$. We then look at the elements of $\pi^i((T^n)^m)$ represented by

$$(T^n)^m \xrightarrow{proj^{\beta}_{m,j}} T^j_{\beta} \xrightarrow{p_j} S^j \xrightarrow{g^{\alpha}_{j,i}} S^i; \quad j \leqq m,\ \beta \in \{1, \ldots, \tbinom{n}{j}\}.$$

By means of induction on m we prove:

For all i,n,m in $\boldsymbol{N}$ with $n \geq m$ such that $\pi^i((T^n)^m)$ is a cohomotopy group, i.e. $i \geq)\frac{m}{2}(+1$, the following short exact sequence is split exact,

$0 \to j^*\pi^i((T^n)^m,\ (T^n)^{m-1}) \to \pi^i((T^n)^m) \xrightarrow{i^*} \pi^i((T^n)^{m-1}) \to 0$, and a basis for $\pi^i((T^n)^m) \simeq \pi^i((T^n)^{m-1}) \oplus j^*\pi^i((T^n)^m,\ (T^n)^{m-1})$ is represented by the elements $g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta}_{m,j}$ except that the long exact sequences are not long enough to the left to allow us to decide whether $g^{\alpha}_{2i-2,i} \circ p_{2i-2} \circ \mathrm{proj}^{\beta}_{2i-2,2i-2}$ is homotopic to 0 or not.

The start of the induction is trivial by the long exact sequences.

Let the above be true for m-1. We see that all the elements $g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m-1,j}$ of $\pi^i((T^n)^{m-1})$ where $\beta' \in \{1, \ldots, \binom{n}{j}\}$ and $j \leq m$-1, have trivial extensions to $(T^n)^m$: $g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m,j}$, which proves that $i^* = i^m_i$ is surjective. Further,

$$g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m-1,j} \text{ non} \simeq 0 \Leftrightarrow g^{\alpha}_{j,i} \circ p_j \text{ non} \simeq 0 \Leftrightarrow g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m,j} \text{ non} \simeq 0.$$

From the isomorphisms $\pi^i(S^j) \simeq \pi^i(S^j, s_0) \simeq \pi_j(S^i, s_0)$, $j \leq 2i$-2, we see for $p \in \boldsymbol{Z}$ that $p(g^{\alpha}_{j,i} \circ f) = (pg^{\alpha}_{j,i}) \circ f\colon X \to S^j \to S^i$ when $\pi^i(X)$ and $\pi^i(S^j)$ are cohomotopy groups (f^* is a homomorphism $\pi^i(S^j) \to \pi^i(X)$). From this we see that $pg^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m-1,j}$ and its extension to $(T^n)^m$: $pg^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m,j}$ are zero homotopic at the same time. This and the fact that the elements $g^{\alpha}_{j,i} \circ p_j \circ \mathrm{proj}^{\beta'}_{m-1,j}$ (when not zero homotopic) represent a basis for $\pi^i((T^n)^{m-1})$ gives us a well defined homomorphism $h\colon \pi^i((T^n)^{m-1}) \to \pi^i((T^n)^m)$ such that $i^* \circ h = 1\pi^i((T^n)^{m-1})$. This and the exactness of the long cohomotopy sequence proves that the sequence above is split exact. Thus

$$\pi^i((T^n)^m) \simeq \pi^i((T^n)^{m-1}) \oplus j^*\pi^i((T^n)^m,\ (T^n)^{m-1}),$$

where $j^*\pi^i((T^n)^m),\ (T^n)^{m-1})$ is generated by

$$g_{m,i}^{\alpha} \circ p_m \circ \mathrm{proj}_{m,m}^{\beta''}: (T^n)^m \to T_{\beta''}^m \to S^m \to S^i, \ (\beta'' \in \{1, \ldots, \tbinom{m}{n}\}).$$

If $m = 2i\text{-}2$ we do not know whether $g_{m,i}^{\alpha} \circ p_m \circ \mathrm{proj}_{m,m}^{\beta''} \simeq 0$ or not.

If $m < 2i\text{-}3$, we get from the above that j_i^m is injective, because i_{i-1}^m is surjective and the sequence exact. The following proof is valid also if $m = 2i\text{-}3$. Because all the generators $g_{j,i-1}^{\alpha} \circ p_j \circ \mathrm{proj}_{m-1,j}^{\beta''}: (T^n)^{m-1} \to S^{i-1}$ of $\pi^{i-1}((T^n)^{m-1})$ have extensions to $(T^n)^m$: $g_{j,i-1}^{\alpha} \circ p_j \circ \mathrm{proj}_{m,j}^{\beta''}: (T^n)^m \to S^{i\text{-}1}$, the exactness gives us:

$\delta_{i-1}^m([g_{j,i-1}^{\alpha} \circ p_j \circ \mathrm{proj}_{m-1,j}^{\beta''}]) = 0 \Rightarrow \delta_{i-1}^m = 0 \Rightarrow j_i^m$ injective, i.e. for $m < 2i\text{-}2$ we have $\pi^i((T^n)^m) \simeq \pi^i((T^n)^{m-1}) \oplus \pi^i((T^n)^m, (T^n)^{m-1})$, and a basis for $\pi^i((T^n)^m)$ is represented by all the

$$g_{j,i}^{\alpha} \circ p_j \circ \mathrm{proj}_{m,j}^{\beta} \quad (\beta \in \{1, \ldots, \tbinom{n}{j}\};\ j \leq m).$$

Theorem 5. *The cohomotopy groups $\pi^i((T^n)^m)$ are finitely generated and for $2i \geq m+3$ we have*

$$\pi^i((T^n)^m) \simeq \begin{cases} 0 \text{ if } i > m \\ \pi_i(S^i, s_0)^{\binom{n}{i}} \oplus \pi_{i+1}(S^i, s_0)^{\binom{n}{i+1}} \oplus \ldots \oplus \pi_m(S^i, s_0)^{\binom{n}{m}} \text{ if } i \leqq m \end{cases}$$

and the last expression is valid for $m = 2i\text{-}2$, if the last factor is replaced by a suitable factor group.

For $m = n$ we get:

Theorem 6. *For $2i \geqq n+3$ we have*

$$\pi^i(T^n) \simeq \begin{cases} 0 \text{ if } i > n \\ \pi_i(S^i, s_0)^{\binom{n}{i}} \oplus \pi_{i+1}(S^i, s_0)^{\binom{n}{i+1}} \oplus \ldots \oplus \pi_n(S^i, s_0) \text{ if } i \leqq n \end{cases}$$

and the last expression is valid for $n = 2i\text{-}2$ if the last factor is replaced by a suitable factor group.

It is known that $\pi_m(S^n, s_0)$ is zero for $m < n$, $\boldsymbol{Z}$ for $m = n$ and a finite abelian group for $m > n$, except $\pi_{4i-1}(S^{2i}, s_0)$ which is the direct sum of $\boldsymbol{Z}$ and a finite group. This group is not a cohomotopy group and so it has no influence on the cohomotopy groups of $(T^n)^m$. Hence:

Corollary. *$\pi^i(T^n)$ is the direct sum of $\boldsymbol{Z}^{\binom{n}{i}}$ and a finite abelian group, when $i \geq)\frac{1}{2}n(+ 1$ (i.e. rank $\pi^i(T^n) = \binom{n}{i}$).*

Chapter 3

Elementary Properties of Tori

Before I discuss the problems of reducibility mentioned in the introduction, I deduce a few elementary results:

Lemma 1. *The homotopy classes of homeomorphisms* $(T^n,t_0) \to (T^n,t_0)$ *are in one to one correspondence with the unimodular* $n \times n$ *matrices.*

Proof. [A unimodular $n \times n$ matrix $\boldsymbol{A}$ has elements in the integers $\boldsymbol{Z}$ and determinant ± 1]. The lemma follows easily from the isomorphism F_n: $[T^n, T^n] \to \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(T^n,t_0)) \simeq \mathrm{Hom}(\boldsymbol{Z}^n,\boldsymbol{Z}^n) \simeq \boldsymbol{Z}^{n^2}$ defined by $F_n[f] = f_*$.

Lemma 2. *The homotopy classes in* T^n *of* T^m*'s imbedded in* T^n *for which* $T^n = T^m \times X$ *are in one to one correspondence with the direct summands of* $\pi_1(T^n,t_0)$ *of rank* m.

This follows easily from Lemma 1.

Theorem 7. f: $T^n \to X$ *is m-reducible iff* f *is homotopic to the projection of* T^n *onto a* T^m *imbedded in* $T^n = T^m \times Y$ *(possibly after a shift of coordinates in* T^n*), followed by a map* $T^m \to X$.

Proof. Obviously, "if" is trivial. We shall prove "only if". It is enough to show that every map f: $(T^n,t_0) \to (T^m,t_0)$ can be projected through a T_1^m imbedded in T^n. Because $\pi_1(T^m,t_0)$ is free, $f_*(\pi_1(T^n,t_0))$ is also free and so $f_*(\pi_1(T^n,t_0))$ is a direct summand in $\pi_1(T^n,t_0)$ of rank $m' \le m$. Let $G \subseteq \pi_1 (T^n,t_0)$ be a direct summand in $\pi_1(T^n,t_0)$ of rank m including the isomorphic image of $f_*(\pi_1(T^n,t_0))$. From Lemma 2 G corresponds to a T_1^m imbedded in $T^n = T_1^m \times Y$ (possibly after a shift of coordinates of T^n). Thus f is homotopic to the projection in new coordinates of T^n onto T_1^m followed by the map $f|T_1^m$: $T_1^m \to T^m$.

Chapter 4

Remarks about Finitely Generated Abelian Groups

We know that there is an isomorphically unique decomposition of a finitely generated abelian group G as a direct sum of cyclic groups $\boldsymbol{Z}^n \oplus \boldsymbol{Z}n_1 \oplus \ldots \oplus \boldsymbol{Z}n_p$, where n_i divides n_{i-1}.

The number n is the rank of G and the n_i's are called the torsion coefficients of G.

We know that every subgroup of a free abelian group $F \simeq \boldsymbol{Z}^m$ of rank m is a free abelian group of rank $m' \leq m$. A finite abelian group G' has in general many decompositions as a direct sum of cyclic groups, but we can define a dimension of a finitely generated abelian group G as

dim G = the smallest number of generators of G.

The usual proof of the theorem above starts with m generators $g_1, \ldots, g_m$ of G and then shows that there exists an $r \leq m$ and $\varepsilon_1, \ldots, \varepsilon_r$ with $\varepsilon_i | \varepsilon_{i-1}$ so that G is isomorphic to $\boldsymbol{Z}^{m-r} \oplus \boldsymbol{Z}_{\varepsilon_1} \oplus \ldots \oplus \boldsymbol{Z}_{\varepsilon_r}$, where $\boldsymbol{Z}_1 = 0$. This gives us that

dim G = rank G + the number of torsion coefficients of G.

The dimension of G has the following properties:

1) $\dim G = m \Rightarrow \forall q \leq m \ \exists G' \subseteq G$: $\dim G' = q$.
2) If $f \in \mathrm{Hom}(\boldsymbol{Z}^n, \tilde{G})$, $\tilde{G}$ a group, then $f(\boldsymbol{Z}^n)$ is a finitely generated abelian group with $\dim f(\boldsymbol{Z}^n) \leq n$.
3) $G' \subseteq G \Rightarrow \dim G' \leq \dim G$.

Chapter 5
Torus Reducibility Problems

Let $X \neq \emptyset$ be a topological space, $x_0 \in X$ a fixed point of X. Looking at torus maps $T_n \to X$, we always assume them, as already mentioned, continuous, and if X is path connected, we assume that t_0 is mapped into x_0. We are interested only in the homotopy classes of maps $T^n \to X$. We know that $\pi_0(X, x_0)$ is the set of path components of X with the path component including x_0 as 0-element, that, for all $i \geq 1$ and all $x_0, x_1 \in X$ in the same path component, $\pi_i(X, x_0)$ is isomorphic to $\pi_i(X, x_1)$ and that T^n itself is path connected. Hence a theorem about torus mappings into path connected spaces under some conditions on the $\pi_i(X, x_0)$'s for fixed x_0 can be translated to a theorem about torus mappings into spaces not necessarily path connected, under the same conditions on the $\pi_i(X, x_1)$'s for all $x_1 \in X$. In the following I therefore assume the spaces X path connected even if this is not written explicitely.

We start with some trivial remarks concerning the definitions 1-3 in the introduction:

For f: $(T^n, t_0) \to (X, x_0)$ we have: If f *is* m-reducible, then $\dim f_*(\pi_1(T^n, t_0)) \leq m$ and $\dim f_{\#p}(H_p(T^n)) \leq \binom{m}{p}$ for every $p \in \boldsymbol{N}$, where

$$f_* \in \mathrm{Hom}(\pi_1(T^n, t_0),\ \pi_1(X, x_0)),\ f_{\#p} \in \mathrm{Hom}(H_p(T^n),\ H_p(X))$$

and

$$\pi_1(T^n, t_0) \simeq \boldsymbol{Z}^n,\ H_p(T^n) \simeq \boldsymbol{Z}^{\binom{n}{p}}.$$

The map F_{m+1}: $[T^{m+1},X] \to \mathrm{Hom}(\pi_1(T^{m+1},t_0),\ \pi_1(X,x_0))/\sim$ defined by $F_{m+1}[f] = \{f_*\}$ is surjective iff every homomorphism h: $\pi_1(T^{m+1},t_0) \to \pi_1(X,x_0)$ is induced by some map f: $(T^{m+1},t_0) \to (X,x_0)$. From Theorems 1 and 3 it follows that this happens when X is an m-space of type $(\pi,1)$ and when X is an H-space.

Theorem 8. *If F_{m+1} is surjective, and X is [n dimensionally $(n > m)$] m-reducible, then every finitely generated abelian subgroup of $\pi_1(X,x_0)$ has dimension $\leqq m$.*

Remark. If $\pi_1(X,x_0)$ is itself a finitely generated abelian group, then the above condition is equivalent to $\dim \pi_1(X,x_0) \leq m$.

Proof of Theorem 8. Suppose $\pi_1(X,x_0)$ has a finitely generated abelian subgroup of dimension greater than m, then $\pi_1(X,x_0)$ also has a subgroup G of dimension $m+1$. Because $\pi_1(T^{m+1},t_0) \simeq \boldsymbol{Z}^{m+1}$, we have a surjective $h \in \mathrm{Hom}(\pi_1(T^{m+1},t_0),\ G)$, which because of the assumption corresponds to an f: $T^{m+1} \to X$ with $f_* = h$. If we define f' as $T^n \xrightarrow{proj} T^{m+1} \xrightarrow{f} X$, then $\dim f'_* \pi_1(T^n,t_0) = \dim h(\pi_1(T^{m+1},t_0)) = m+1$. Thus f' is not m-reducible.

Theorem 9. *Suppose F_n: $[T^n,X] \to \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(X,x_0))/\sim$ defined by $F_n[f] = \{f_*\}$ is bijective. (From Theorem 1 we know that this is the case when X is an n-space of type $(\pi_1,1)$). Then*

$$f\colon (T^n,t_0) \to (X,x_0) \text{ is } m\text{-reducible iff } f_*(\pi_1(T^n,t_0))$$

(an abelian group of dimension $\leqq n$) is of dimension $\leqq m$.

Proof. We have already proved "only if". If $\dim f_*\pi_1(T^n,t_0) \leq m$, then f_* can be factorized through $\pi_1(T^m,t_0) \simeq \boldsymbol{Z}^m$ so that we can choose $\tilde{h} \in \mathrm{Hom}\,(\pi_1(T^n,t_0),\ \pi_1(T^m,t_0))$ and $\tilde{g} \in \mathrm{Hom}\,(\pi_1(T^m,t_0),\ \pi_1(X,x_0))$ such that $f_* = \tilde{g} \circ \tilde{h}$. Because F_n surjective implies that F_m is surjective for $m \leqq n$ there exists a g: $T^m \to X$ with $g_* = \tilde{g}$, We also have an h: $T^n \to T^m$ with $h_* = \tilde{h}$. Thus $g_* \circ h_* = f_*$ or $g \circ h \simeq f$, i.e. f is m-reducible.

This gives us the following theorems:

Theorem 10. *Let F_n be bijective. (This is the case when X is an n-space of type $(\pi_1,1)$). Then X is n-dimensionally m-reducible $(1 \leq m < n)$ iff the dimension of any finitely generated abelian subgroup of $\pi_1(X,x_0)$ is $\leqq m$.*

Remark. Instead of "F_n bijective" we could write "F_n injective and F_{m+1} surjective".

Theorem 11. *Let F_n be bijective for all $n > m$. (This is the case when X is a space of type $(\pi_1,1)$). Then X is m-reducible iff the dimension of any finitely generated abelian subgroup of $\pi_1(X,x_0)$ is $\leqq m$.*

We now show

Theorem 12. *X is n-dimensionally 1-reducible $(n > 1)$ iff every finitely generated abelian group of $\pi_1(X,x_0)$ is cyclic and F_n is injective.*

Corollary. *Let $\pi_1(X,x_0) = 0$. Then X is n-dimensionally 1-reducible $(n > 1)$ iff $[T^n,X] = 0$.*

Proof of Theorem 12. Because F_2 is always surjective, it follows from the remark above that it only remains to show that F_n injective is a necessary condition.

Let $f,g\colon T^n \to X$ be given such that $f_* \sim g_*$ (i.e. $f|(T^n)^1 \simeq g|(T^n)^1$). We suppose that X is n-dimensionally 1-reducible and then want to show that $f \simeq g$. Because f and g are 1-reducible and

$$[T^n,S^1] \simeq \mathrm{Hom}(\pi_1(T^n,t_0),\ \pi_1(S^1,s_0)),$$

$$[S^1,X] \simeq \mathrm{Hom}(\pi_1(S^1,s_0),\ \pi_1(X,x_0))/\sim,$$

we can choose the factorizations $T^n \overset{h_1}{\to} S^1 \overset{g_1}{\to} X$ and $T^n \overset{h_2}{\to} S^1 \overset{g_2}{\to} X$ of f and g such that $g_{1*}\pi_1(S^1,s_0) = f_*\pi_1(T^n,t_0)$ and $g_{2*}(\pi_1(S^1,s_0)) = g_*\pi_1(T^n,t_0)$ and $h_{1*} = h_{2*}$. Then $h_1 \simeq h_2$ and $g_1 \simeq g_2$. Thus

$$f \simeq g_1 \circ h_1 \simeq g_2 \circ h_2 \simeq g.$$

This gives us

Theorem 13. *X is 1-reducible iff every finitely generated abelian subgroup of $\pi_1(X,x_0)$ is cyclic and every F_n is injective.*

Theorem 14. *Let X be a p-space of type (π_p,p) with π_p abelian. Then we have:*

1) If $m \leq n < p$, we have $[T^n,X] \simeq [T^m,X] \simeq 0$ so X is trivially n-dimensionally m-reducible.

2) If $m < p \leqq n$, we have $[T^m,X] = 0$, but $[T_n,X]$ is non trivial if π_p is non trivial, so a necessary condition for X to be n-dimensionally m-reducible is that $\pi_p = 0$.

3) *If* $p \leqq m < n$ *and* $G_{m+1}\colon [T^{m+1},X] \to \mathrm{Hom}(H_p(T^{m+1}), \pi_p)$ *defined by* $G_{m+1}[f] = \varkappa^{-1} f_{\#p}$ *is surjective (which is the case when* X *is an* m*-space of type* (π_p,p) *or when* X *is an* H*-space of* p*-type* (π_p,p)*), then a necessary condition for* X *to be* n*-dimensionally* m*-reducible is that the dimension of any finitely generated subgroup of* π_p *is* $\leqq \binom{m}{p} = \dim H_p(T^m)$.

The proof of 3) is analogous to the proof of Theorem 8 because of the remark p. 17.

Corollary. *Let* X *be a* p*-space of type* (π_p,p) *with* π_p *abelian. Then we have:*

1) *For* $m < p$ *a necessary condition for* X *being* m*-reducible is* $\pi_p = 0$.

2) *For* $m \geq p$ *and* G_{m+1} *surjective a necessary condition for* X *being* m*-reducible is that the dimension of any finitely generated subgroup of* π_p *is* $\leqq \binom{m}{p}$.

Examples. 1) Because the fundamental groups of the complex and quaternionic projective spaces are zero but not all the homotopy groups are zero, these spaces are not 1-reducible.

2) $S^n (n > 1)$ is not m-reducible for $m < n$.

3) $P^n (n > 1)$ is not m-reducible for $m < n$:

We have the covering projection $p\colon S^n \to P^n$ and a map $p_n\colon T^n \to S^n$ which is not 0-homotopic. p_n is not m-reducible, nor is $p_n \circ (\times N)_n\colon T^n \to T^n \to S^n$ m-reducible. We now look back at $p \circ p_n\colon T^n \to S^n \to P^n$. The map $p \circ p_n \circ (\times N)_n$ is not 0-homotopic since p is a covering projection. If $p \circ p_n$ were m-reducible for some $m < n$, i. e. $p \circ p_n \simeq g \circ h\colon T^n \xrightarrow{h} T^m \xrightarrow{g} P^n$, then, since $[T^m,P^n] \simeq \mathrm{Hom}(\boldsymbol{Z}^m,\boldsymbol{Z}_2)$, we would obtain

$$p \circ p_n \circ (\times 2)_n \simeq g \circ h \circ (\times 2)_n = [g \circ (\times 2)_m] \circ h \simeq 0 \circ h = 0$$

(we assume h linear because $[T^n,T^m] \simeq \mathrm{Hom}(\pi_1(T^n,t_0), \pi_1(T^m,t_0))$, contradicting $p \circ p_n \circ (\times 2)_n$ non $\simeq 0$. Thus $p \circ p_n$ is not m-reducible.

4) From the computation of the cohomotopy groups of T^n and the results in this chapter one can get further results about the reducibility of spheres; for instance:

If $2i \geq n+3$, $i < n$ and $\pi_n(S^i,s_0) \neq 0$ then we get from Theorem 6 that the map $g^{\alpha}_{n,i} \circ p_n\colon T^n \to S^n \to S^i$ with $g^{\alpha}_{n,i}$ non $\simeq 0$ is not homotopic to zero. This map f is algebraically trivial, i.e. its induced homomorphisms between the homology and cohomology groups are trivial. If f were i-reducible, then

Theorem 7 would give that f could be projected through a $T^i(T^n = T^i \times T^{n-i}$, perhaps after a shift of coordinates). Thus f would be homotopic to a composed map

$$T^n \xrightarrow{h} T^n \xrightarrow{proj} T^i \xrightarrow{g} S^i,$$

where g non $\simeq 0$, and h is a change of basis. Because $[T^i,S^i] \simeq \mathrm{Hom}(H_i(T^i), H_i(S^i)) \simeq \mathrm{Hom}(\boldsymbol{Z},\boldsymbol{Z})$, $g_{\#i} \neq 0$ and so $g \circ \mathrm{proj} \circ h$ would not be algebraically trivial, contradicting $f \simeq g \circ \mathrm{proj} \circ h$. Thus if $2n > 2i \geq n+3$ and $\pi_n(S^i,s_0) \neq 0$, then S^i is not i-reducible.

5) In an analogous way we get:

S^2 is not 2-reducible,

because the map $T^3 \xrightarrow{p_3} S^3 \xrightarrow{p} S^2$, where p is the Hopf fibration, is known to be non $\simeq 0$ when p_3 non $\simeq 0$.

Chapter 6
Almost Periodic Movements

Definition. *A topological space X is called continuously locally arcwise connected when to every compact subspace K of X there exists a neighbourhood $O \subseteq K \times K$ of the diagonal Δ_K in $K \times K$ and a continuous map Φ: $O \times I \to X$ so that $\Phi(x,x,t) = x$, $\Phi(x,y,0) = x$, $\Phi(x,y,1) = y$.*

For metric spaces this is equivalent to the definition (for metric spaces only) used in [4].

Remarks. 1) When X is a metric space for which any two points x and y whose distance remains below a certain number can be connected by a geodetic arc which depends continuously on x and y, then X is continuously locally arcwise connected. 2) Any CW-complex is continuously locally arcwise connected. Indeed it satisfies the following:

There exists a covering $(U_j | j \in J)$ of X with open sets and a continuous function Φ: $O \times I \to X$, where $O = \bigcup_J U_j \times U_j$, such that $\Phi(x,x,t) = x$, $\Phi(x,y,0) = x$, $\Phi(x,y,1) = y$. This can be shown by induction: Assuming (U_j) chosen and Φ constructed on the n-1 skeleton X^{n-1} of X, it can be shown that the definitions can be extended to X^n, which is obtained from X^{n-1} by adjoining n-cells and which has the topology coherent with X^{n-1} and the n-cells.

Definition. *A continuous movement in a metric space X is a continuous function $x = f(t)$, $t \in \boldsymbol{R}$, $x \in X$. A number $\tau = \tau_f(\varepsilon)$ is called a translation* *number of* $f(t)$ *corresponding to* $\varepsilon > 0$ *if the condition dist* $(f(t), f(t+\tau)) \leq \varepsilon$ *is satisfied for all* $t \in \boldsymbol{R}$. *The movement* $x = f(t)$ *is called almost periodic if the range of* $f(t)$ *lies in a compact subset of* X *and the set* $\{\tau_f(\varepsilon)\}$ *is relatively dense for every* $\varepsilon > 0$ (i.e. *there exists an* $l > 0$ *so that every interval of length* l *contains at least one of the* $\tau_f(\varepsilon)$*'s*).

Definition. *A function* $x = f(t,v)$; $t \in \boldsymbol{R}$, $v \in [0,1]$, $x \in X$; *is called a uniformly continuous family of almost periodic movements when* 1) *the range of* $f(t,v)$ *lies in a compact subset of* X, 2) *for all* $v_0 \in [0,1]$ $f(t,v_0)$ *is almost periodic,* 3) *to all* $\varepsilon > 0$ *and all* $v_0 \in [0,1]$ *corresponds a neighbourhood* $U_\varepsilon(v_0)$ *of* v_0 *such that for all* $t \in \boldsymbol{R}$, *all* $v \in U_\varepsilon(v_0)$: *dist*$(f(t,v_0), f(t,v)) \leq \varepsilon$.

Remark. Tornehave's definitions are the same except that he does not demand that the ranges lie in compact subsets. Instead he is mainly interested in complete metric spaces and he shows that the closure of the range of an almost periodic movement $f(t)$ in a complete metric space X is a compact subspace. In the same way it can be shown that the closure of the range of a uniformly continuous family of a.p. movements in a complete metric space is a compact subspace. This gives us that the definitions coincide for complete metric spaces, and this is all I need. With the new definitions Tornehave's results about complete metric spaces can easily be extended to arbitrary metric spaces. The new definition can be extended further to arbitrary topological spaces because it is possible to introduce one and only one uniformity structure on compact spaces.

Let $\mathscr{C}$ denote the class of metric spaces which are continuously locally arcwise connected. Then $\mathscr{C}$ includes the class $\mathscr{C}'$ of metrizable CW complexes, which again includes the class $\mathscr{C}''$ of locally compact polyhedrons.

Definition. *Two almost periodic movements* $x_1 = f_1(t)$ *and* $x_2 = f_2(t)$ *are called a(lmost) p(eriodically) homotopic iff there exists a uniformly continuous family* $x = f(t,v)$ *with* $f(t,0) = f_1(t)$ *and* $f(t,1) = f_2(t)$.

This relation is obviously an equivalence relation in the set of a.p. movements in X and it leads to a subdivision of this set into a.p. homotopy classes.

Remark. A continuous function $\tilde{f}: \boldsymbol{R}^n \to X$ which is periodic in all the variables with the same period $r \in \boldsymbol{R}$ (called a torus map by Tornehave) corresponds to torus maps $T^n \to X$ (we can look at T^n as $\boldsymbol{R}^n/(qr\boldsymbol{Z})^n$, $q \in \boldsymbol{Z}$). The function $g: \boldsymbol{R}^n \to X$: $g(t) = \tilde{f}(pt)$, $p \in \boldsymbol{R}$ has period $\frac{r}{p}$ and corresponds to the same torus maps as $\tilde{f}$ (we can look at T^n as $\boldsymbol{R}^n/(\frac{qr}{p}\boldsymbol{Z})^n$, $q \in \boldsymbol{Z}$). Usually

I think of T^n as $\boldsymbol{R}^n/\boldsymbol{Z}^n$ and I call the variables $(u_1, \ldots, u_n)$ with $u_i \in [0,1]$. If $(\beta_1, \ldots, \beta_n)$ are real numbers and t is also real, and $f\colon T^n \to X$ is a map, I write $f(\beta_1 t, \ldots, \beta_n t)$ for $f(\{\beta_1 t\}, \ldots, \{\beta_n t\})$, where $\{\beta t\}$ is the (one of the) $u \in [0,1]$ for which $\beta t \equiv u \pmod 1$. It is well known that $x = f(\beta_1 t, \ldots, \beta_n t)$ is an almost periodic movement in X. In the opposite direction we get from [4], p. 28:

Theorem. *Every almost periodic movement $x_1 = f(t)$ in $X \in \mathscr{C}$ is a.p. homotopic to a certain $x_2 = g(\beta_1 t, \ldots, \beta_n t)$ with $g\colon T^n \to X$ and $(\beta_1, \ldots, \beta_n)$ rationally independent real numbers* (i.e. *independent as vectors in a* $\boldsymbol{Q}$*-vector space*).

The following theorem is [4] lemma 22 with a correction[1]).

Theorem 15. *Let $\tilde{f}(t)$ and $\tilde{g}(t)$ be two almost periodic movements in $X \in \mathscr{C}$. In order to investigate whether $\tilde{f}$ and $\tilde{g}$ are a.p. homotopic we choose a common set of n rationally independent real numbers $(\beta_1', \ldots, \beta_p')$ and torus maps $f', g'\colon T^p \to X$ such that $\tilde{f}(t)$ is a.p. homotopic to $f'(\beta_1' t, \ldots, \beta_p' t)$ and $\tilde{g}(t)$ is a.p. homotopic to $g'(\beta_1' t, \ldots, \beta_p' t)$. Then $\tilde{f} \underset{\text{a.p.}}{\simeq} \tilde{g}$ iff there exists a natural number N, such that $f' \circ (\times N) \simeq g' \circ (\times N)$, where $\times N\colon T^p \to T^p$ is defined by $(\times N)(u_1, \ldots, u_p) = (Nu_1, \ldots, Nu_p)$.*

I shall now discuss the notions introduced in definitions 4–7 in the introduction.

Theorem 16. *Let $X = \tilde{f}(t)$ be an a.p. movement in $X \in \mathscr{C}$. Let $f\colon T^n \to X$ be one of its corresponding torus maps* (i.e. *$f(\beta_1 t, \ldots, \beta_n t) \underset{\text{a.p.}}{\simeq} \tilde{f}(t)$ and $(\beta_1, \ldots, \beta_n)$ are rationally independent*). *Then $\tilde{f}$ is a.p. m-reducibel iff $f \circ (\times N)$ is m-reducible for some natural number N.*

Proof. We may assume $m < n$. We shall first prove "only if". Let $\tilde{f}$ be a.p. m-reducible. We choose $g\colon T^m \to X$ and rationally independent numbers $(\gamma_1, \ldots, \gamma_m)$ such that

$$g(\gamma_1 t, \ldots, \gamma_m t) \underset{\text{a.p.}}{\simeq} \tilde{f}(t) \underset{\text{a.p.}}{\simeq} f(\beta_1 t, \ldots, \beta_n t).$$

We now look at the vector space V over $\boldsymbol{Q}$ spanned by $(\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m)$. Then $p = \dim V \geq \max\{m,n\}$. We supplement $(\beta_1, \ldots, \beta_n)$ to a basis $(\beta_1', \ldots, \beta_n', \beta_{n+1}', \ldots, \beta_p')$ with $\beta_\nu' = \beta_\nu$, $\nu = 1, \ldots, n$. Then the γ_i's are rational linear combinations of (β_j'). Thus there exists a natural number M

[1]) Tornehave told me about this mistake. He overlooked the possibility of the factor $(\times N)$ such that his condition is too strong. The examples proving the existence of non-trivial a.p. movements are never the less correct, because the relevant obstructions belong to infinite cyclic groups.

so that the $M\gamma_i$'s are integral linear combinations Λ_i, $i = 1, \ldots, m$ of (β_j'). We now define a map $f_1\colon T^p \to T^n \to X$ by $f_1(u_1, \ldots, u_n, u_{n+1}, \ldots, u_p) = f(Mu_1, \ldots, Mu_n)$ and a map $g_1\colon T^p \to T^m \to X$ by $g_1(u_1, \ldots, u_p) = g(\Lambda_1(\boldsymbol{u}), \ldots, \Lambda_m(\boldsymbol{u}))$.

Because $f(\beta_1 t, \ldots, \beta_n t) \underset{\text{a.p.}}{\simeq} g(\gamma_1 t, \ldots, \gamma_m t)$ we have

$$f(M\beta_1 t, \ldots, M\beta_n t) = f_1(\beta_1' t, \ldots, \beta_p' t) \underset{\text{a.p.}}{\simeq} g(M\gamma_1 t, \ldots, M\gamma_m t) = g_1(\beta_1' t, \ldots, \beta_p' t)$$

(and $(\beta_1', \ldots, \beta_p')$ are rationally independent). Theorem 15 tells us that there exists a natural number N_1 such that $f_1 \circ (\times N_1) \simeq g_1 \circ (\times N_1)$. Then the restrictions to the $T^n \subseteq T^p$ corresponding to $u_{n+1} = \ldots = u_p = 0$ are homotopic, i.e.

$$f_1(N_1u_1, \ldots, N_1u_n, 0, \ldots, 0) = f(MN_1u_1, \ldots, MN_1u_n) \simeq g_1(N_1u_1, \ldots, N_1u_n, 0, \ldots, 0)$$

$$= g(N_1\Lambda_1(u_1, \ldots, u_n, 0, \ldots, 0), \ldots, N_1\Lambda_m(u_1, \ldots, u_n, 0, \ldots, 0)).$$

Denoting by $h\colon T^n \to T^m$ the map defined by

$$(u_1, \ldots, u_n) \to (N_1\Lambda_1(u_1, \ldots, u_n, 0, \ldots, 0), \ldots, N_1\Lambda_m(u_1, \ldots, u_n, 0, \ldots, 0))$$

we have $f \circ (\times N_1 M) \simeq g \circ h$, where $h\colon T^n \to T^m$ and $g\colon T^m \to X$, so that $f \circ (\times N_1 M)$ is m-reducible.

Next, we shall prove "if". Suppose $f' = f \circ (\times N)$ is m-reducible. We have $f'(\beta_1' t, \ldots, \beta_n' t) \underset{\text{a.p.}}{\simeq} \tilde{f}(t)$, where $(\beta_1' = \beta_1/N, \ldots, \beta_n' = \beta_n/N)$ are rationally independent. Because $[T^n, T^m] \simeq \mathrm{Hom}(\pi_1(T^n, t_0), \pi_1(T^m, t_0))$ we can choose $h\colon T^n \to T^m$ linear (corresponding to $\boldsymbol{A}_{m,n}$) and $g\colon T^m \to X$ so that $f' \simeq g \circ h$. If we put

$$\begin{pmatrix} \tilde{\beta}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\beta}_m \end{pmatrix} = \boldsymbol{A}_{m,n} \begin{pmatrix} \beta_1' \\ \cdot \\ \cdot \\ \cdot \\ \beta_n' \end{pmatrix},$$

then

$$\tilde{f}(t) \underset{\text{a.p.}}{\simeq} f'(\beta_1' t, \ldots, \beta_n' t) \underset{\text{a.p.}}{\simeq} g \circ h(\beta_1' t, \ldots, \beta_n' t) = g(\tilde{\beta}_1 t, \ldots, \tilde{\beta}_m t),$$

so that $\tilde{f}$ is a.p. m-reducible.

Remarks. 1) $(f \circ (\times N))_* = Nf_*$, so rank $f_*\pi_1(T^n, t_0) = r$ iff there exists a natural number N such that $(f \circ (\times N))_*\pi_1(T^n, t_0) \simeq \boldsymbol{Z}^r$.

2) $(f \circ (\times N))_{\#p} = N^p f_{\#p}$, so rank $f_{\#p}\, H_p(T^n) = r$ iff there exists a natural number N such that $(f \circ (\times N))_{\#p}\, H_p(T^n) \simeq \boldsymbol{Z}^r$.

Statement 2) follows immediately from the expression of elements of $H_p(T^n)$ as cross products of elements from $H_1(S^1)$. The remarks and Theorem 16 gives us that the theorems from chapter V can be changed to theorems about almost periodic movements in $X \in \mathscr{C}$, by translating "dim" to "rank". Let $\mathscr{C}_0$ denote the class of path connected spaces in $\mathscr{C}$.

Remark. Let $x = \tilde{f}(t)$ be an a.p.-movement in $X \in \mathscr{C}_0$ corresponding (for some $(\beta_1, \ldots, \beta_n)$) to $f\colon T^n \to X$. If $\tilde{f}(t)$ is a.p. m-reducible, then rank $f_* \pi_1(T^n, t_0) \leq m$ and rank $f_{\#p}\, H_p(T^n) \leq \binom{m}{p}$ for every p.

Theorem 17. *Suppose that* $F_{m+1}\colon [T^{m+1}, X] \to \mathrm{Hom}(\pi_1(T^{m+1}, t_0), \pi_1(X, x_0))/\sim$ *is surjective and that* $X \in \mathscr{C}_0$ (e.g. *X is an m-space of type* $(\pi_1, 1)$ *or an H-space*). *If, further, X is a.p. m-reducible, then the rank of any abelian subgroup of* $\pi_1(X, x_0)$ *is* $\leqq m$.

Theorem 18. *Suppose that* $F_n\colon [T^n, X] \to \mathrm{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0))/\sim$ *is bijective and that* $X \in \mathscr{C}_0$ (e.g. *X is an n-space of type* $(\pi_1, 1)$). *Then an almost periodic movement corresponding to* $f\colon T^n \to X$ *is a.p. m-reducible iff* rank $f_*(\pi_1(T^n, t_0)) \leqq m$.

Theorem 19. *Let* $F_n\colon [T^n, X] \to \mathrm{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0))/\sim$ *be bijective for every* $n > m$ (e.g. *X is a space of type* $(\pi_1, 1)$), *and let* $X \in \mathscr{C}_0$. *Then X is a.p. m-reducible iff the rank of every abelian subgroup of* $\pi_1(X, x_0)$ *is* $\leqq m$.

Theorem 20. *A space* $X \in \mathscr{C}_0$ *is a.p.* 1*-reducible* (i.e. *every a.p. movement in X is a.p. homotopic to a periodic movement*) *iff the rank of every abelian subgroup of* $\pi_1(X, x_0)$ *is* $\leqq 1$ *and for every natural number n every pair of maps* $f, g\colon T^n \to X$ *satisfying* $F_n[f] = F_n[g]$ *also satisfies the condition* $f \circ (\times N) \simeq g \circ (\times N)$ *for some natural number N.*

Theorem 21, 22. *Let* $X \in \mathscr{C}_0$ *be a p-space of type* (π_p, p) *with* π_p *abelian. Then*

1) *if X is a.p. m-reducible for some* $m < p$, *then* rank $\pi_p = 0$.

2) *if* $p \leqq m$ *and* $G_{m+1}\colon [T^{m+1}, X] \to \mathrm{Hom}(H_p(T^{m+1}), \pi_p)$ *defined by* $G_{m+1}[f] = \varkappa^{-1} f_{\#p}$ *is surjective* (e.g. *X is an m-space of type* (π_p, p) *or X is an H-space of p-type* (π_p, p)) *and X is a.p. m-reducible, then* rank $\pi_p \leqq \binom{m}{p}$.

Examples. 1) Because the fundamental groups of the complex and quaternionic projective spaces are zero and the first nontrivial homotopy groups are isomorphic to $\boldsymbol{Z}$, these spaces are not a.p. 1-reducible.

2) S^n $(n > 1)$ is not a.p. m-reducible for $m < n$ $(\pi_n(S^n,s_0) \simeq \boldsymbol{Z})$.

3) P^n $(n > 1)$ is not a.p. m-reducible for $m < n$ (proof analogous to the proof p. 20).

4) S^2 is not a.p. 2-reducible $([T^3,S^3] \simeq \boldsymbol{Z})$.

Bibliography

[1] R. H. Fox: *Homotopy Groups and Torus Homotopy Groups.* Ann. of Math. (2) 49 (1948), p. 371–510.

[2] Sze-Tsen Hu: *Homotopy Theory.* New York 1959.

[3] Edwin H. Spanier: *Algebraic Topology.* New York 1966.

[4] Hans Tornehave: *On Almost Periodic Movements.* Mat. Fys. Medd. Dan. Vid. Selsk. **28,** 13 (1954).

Indleveret til Selskabet den 15. jan. 1971.
Færdig fra trykkeriet den 7. april 1972.